Int. J. Solids Structures Vol. 17, pp. 291–299
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0020-7683/81/0301-0291/$02.00/0

A UNIQUENESS THEOREM IN NONLINEAR VISCOELASTICITY WITH APPLICATION TO TEMPERATURE AND IRRADIATION INDUCED CREEP PROBLEMS

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(*Received* 8 *September* 1979; *in revised form* 13 *June* 1980)

Abstract—A uniqueness theorem is first derived for a constitutive relation in the form of a nonlinear memory integral with aging included. Uniqueness is proved for the solution to the dynamic mixed boundary value problem with small deformations. The theorem is then specialized to a constitutive equation of the isotropic power law type governing thermoirradiation induced creep.

1. INTRODUCTION

Uniqueness theorems of solutions for infinitesimal creep in linear viscoelastic materials have been given by Gurtin and Sternberg[1], Edelstein and Gurtin[2], Odeh and Tadjbakhsh[3], Lubliner and Sackman[4], and others. Sackman also gave theorems for nonlinear Maxwell materials[5] and for materials undergoing nonlinear infinitesimal quasistatic steady creep with elastic strain ignored[6]. Recently Edelstein gave uniqueness theorems for nonlinear creep with strain hardening included[7] for a special domain and loading, and with the strain hardening excluded[8] for more general domains and loading. In[9], a uniqueness theorem was given for an isotropic nonlinear constitutive relation for thermal creep, which includes elastic strain, transient creep, material aging and creep compressibility. Most recently, Gurtin, Reynolds and Spector investigated the questions of uniqueness and stability in quasi-static nonlinear viscoelasticity[10]. In the present paper we prove uniqueness for a constitutive relation which includes the same effects as in[9] plus the additional effects of thermal expansion, irradiation swelling, thermoirradiation induced creep, and temperature and flux dependent material properties. This constitutive relation is an extension of the one given in[11] using the time hardening (aging) procedure of[12].

A uniquenes theorem is first derived in Section 2 for a constitutive relation in the form of a nonlinear memory integral with aging included, which is valid for homogeneous isotropic materials characterized by a single creep function. Uniqueness is established for the dynamical mixed boundary value problem assuming small deformations. Then in Section 3 we specialize the uniqueness theorem of Section 2 to the constitutive equation for temperature and irradiation induced creep mentioned above. We also comment on the applicability of the theorem of Section 2 to a wide variety of other constitutive relations given in the literature.

2. A UNIQUENESS THEOREM

Theorem. Let V be an open bounded region in R^3 with regular boundary $\partial V = \partial V_\sigma \cup \partial V_u$, $\partial V_\sigma \cap \partial V_u = \phi$ and let $\bar{V} = V \cup \partial V$. Let $\mathbf{n}$ be the unit outward normal vector to ∂V. Let there be given vector functions $\mathbf{F}(\mathbf{x}, t) \in C(V \times (0, \infty))$, $\mathbf{f}(\mathbf{x}, t) \in C(\partial V_\sigma \times (0, \infty))$, $\mathbf{g}(\mathbf{x}, t) \in C^1(\partial V_u \times (0, \infty))$, $\mathbf{h}(\mathbf{x}) \in C(\bar{V})$ and $\mathbf{k}(\mathbf{x}) \in C(\bar{V})$ satisfying the compatibility conditions

$$\lim_{t \to 0^+} \mathbf{g}(\mathbf{x}, t) = \mathbf{h}(\mathbf{x}), \qquad \mathbf{x} \in \partial V_u,$$

$$\lim_{t \to 0^+} \frac{\partial \mathbf{g}(\mathbf{x}, t)}{\partial t} = \mathbf{k}(\mathbf{x}), \qquad \mathbf{x} \in \partial V_u.$$

Then, there exists at most one set of strains $\epsilon_{ij}(\mathbf{x}, t) \in C^1(\bar{V}\times[0,\infty))$ and stresses $\sigma_{ij}(\mathbf{x}, t) \in C^1(\bar{V}\times[0,\infty))$ satisfying the constitutive relation (possibly nonlinear)†

$$\epsilon_{ij}(\mathbf{x}, t) = \int_0^t J[\sigma_{ij}(\mathbf{x}, t'), t', t]\frac{\partial\sigma_{ij}(\mathbf{x}, t')}{\partial t'}\,\mathrm{d}t', \tag{2.1}$$

in $\bar{V}\times[0,\infty)$, and the field, boundary and initial conditions

$$\sigma_{ij,j} + F_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \quad \text{on} \quad V\times(0,\infty), \tag{2.2}$$

(where $\rho(\mathbf{x}) \in C(\bar{V})$ is the mass density of the medium),

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{in} \quad \bar{V}\times[0,\infty), \tag{2.3}$$

$$\sigma_{ij}n_j = f_i, \quad \text{on} \quad \partial V_\sigma\times(0,\infty), \tag{2.4}$$

$$u_i = g_i, \quad \text{on} \quad \partial V_u\times(0,\infty), \tag{2.5}$$

$$u_i = h_i, \quad \text{in} \quad \bar{V} \quad \text{at} \quad t = 0,‡ \tag{2.6}$$

$$\frac{\partial u_i}{\partial t} = k_i, \quad \text{in} \quad \bar{V} \quad \text{at} \quad t = 0, \tag{2.7}$$

provided J satisfies

$$\lim_{\substack{t'\to t\\ t'<t}} J[\sigma_{ij}(\mathbf{x}, t), t', t] > 0, \tag{2.8}$$

$$\lim_{\substack{t'\to t\\ t'<t}} \frac{\partial J}{\partial t}[\sigma_{ij}(\mathbf{x}, t'), t', t] \geq 0, \tag{2.9}$$

and the continuity conditions

$$J[\sigma_{ij}, t', t] \in C^1(D\times(0,\infty)\times(0,\infty))$$

where D is some appropriate range for stresses containing 0 and

$$\frac{\partial^2 J}{\partial t'\partial t}[\sigma_{ij}(\mathbf{x}, t'), t', t]$$

and

$$\frac{\partial^2 J}{\partial\sigma_{kl}\partial t}[\sigma_{ij}(\mathbf{x}, t'), t', t] \in C(D\times(0,\infty)\times(0,\infty)).$$

Furthermore, there is at most one displacement vector $\mathbf{u}(\mathbf{x}, t) \in C^2(\bar{V}\times[0,\infty))$ defined up to a rigid body displacement.

Proof. The rate of work W of the external forces can be expressed (upon using eqns 2.2 and 2.3 and the divergence theorem§) as

$$\frac{đW}{\mathrm{d}t} = \frac{\mathrm{d}K}{\mathrm{d}t} + \int_V \sigma_{ij}\frac{\partial\epsilon_{ij}}{\partial t}\,\mathrm{d}v¶ \tag{2.10}$$

†We assume that $\epsilon_{ij}(\mathbf{x}, 0) = \sigma_{ij}(\mathbf{x}, 0) = 0$; further discussion of this constitutive relation will be given in the Remarks at the end of the section and in Section 3.

‡Inasmuch as $\epsilon_{ij}(\mathbf{X}, 0) = 0$ (see eqn 2.1), the h_i's cannot be chosen arbitrarily.

§The divergence theorem can be used in view of the regularity assumption on ∂V.

¶The notation $đW/\mathrm{d}t$ is used to indicate that the rate of work is not in general the comoving derivative of an integral.

where

$$K = \frac{1}{2}\int_V \rho \frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t}\,\mathrm{d}v \geq 0$$

is the kinetic energy.

Suppose σ^1_{ij}, ϵ^1_{ij}, u^1_i and σ^2_{ij}, ϵ^2_{ij}, u^2_i are two sets of solutions to eqns (2.1)–(2.7) and consider the difference solution $\sigma_{ij} = \sigma^1_{ij} - \sigma^2_{ij}$, etc. It satisfies eqns (2.2)–(2.7) with $\mathbf{F} \equiv \mathbf{f} \equiv \mathbf{g} \equiv \mathbf{h} \equiv \mathbf{k} \equiv 0$. Therefore, for the difference solution, $\mathrm{d}W/\mathrm{d}t = 0$ and $W(t) = 0$ for all $t \in [0, \infty)$, i.e. integrating (2.10)

$$0 = K(t) + \int_0^t \int_V \sigma_{ij}(\mathbf{x}, t')\frac{\partial \epsilon_{ij}(\mathbf{x}, t')}{\partial t'}\,\mathrm{d}v\,\mathrm{d}t'. \tag{2.11}$$

Substituting (2.1) in (2.11) and using Leibniz' rule (which is justified in view of the continuity assumption on J) we obtain

$$\begin{aligned}0 = K(t) + \int_0^t \int_V \sigma_{ij}(\mathbf{x}, t')\Big\{&J[\sigma^1_{ij}(\mathbf{x}, t'), t', t']\frac{\partial \sigma^1_{ij}(\mathbf{x}, t')}{\partial t'} \\ &- J[\sigma^2_{ij}(\mathbf{x}, t'), t', t']\frac{\partial \sigma^2_{ij}(\mathbf{x}, t')}{\partial t'} + \int_0^{t'}\Big[\frac{\partial J}{\partial t'}[\sigma^1_{ij}(\mathbf{x}, t''), t'', t'] \\ &\times \frac{\partial \sigma^1_{ij}(\mathbf{x}, t'')}{\partial t''} - \frac{\partial J}{\partial t'}[\sigma^2_{ij}(\mathbf{x}, t''), t'', t']\frac{\partial \sigma^2_{ij}(\mathbf{x}, t'')}{\partial t''}\Big]\mathrm{d}t''\Big\}\,\mathrm{d}v\,\mathrm{d}t',\end{aligned}$$

or after regrouping terms (using $\sigma_{ij} = \sigma^1_{ij} - \sigma^2_{ij}$)

$$\begin{aligned}0 = K(t) &+ \int_0^t \int_V \sigma_{ij}(\mathbf{x}, t')J[\sigma^2_{ij}(\mathbf{x}, t'), t', t']\frac{\partial \sigma_{ij}(\mathbf{x}, t')}{\partial t'}\,\mathrm{d}v\,\mathrm{d}t' \\ &+ \int_0^t \int_V \{J[\sigma^1_{ij}(\mathbf{x}, t'), t', t'] - J[\sigma^2_{ij}(\mathbf{x}, t'), t', t']\}\frac{\partial \sigma^1_{ij}(\mathbf{x}, t')}{\partial t'}\sigma_{ij}(\mathbf{x}, t')\,\mathrm{d}v\,\mathrm{d}t' \\ &+ \int_0^t \int_V \int_0^{t'} \sigma_{ij}(\mathbf{x}, t')\frac{\partial J}{\partial t'}[\sigma^2_{ij}(\mathbf{x}, t''), t'', t']\frac{\partial \sigma_{ij}(\mathbf{x}, t'')}{\partial t''}\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t' \\ &+ \int_0^t \int_V \int_0^{t'} \Big\{\frac{\partial J}{\partial t'}[\sigma^1_{ij}(\mathbf{x}, t''), t'', t'] - \frac{\partial J}{\partial t'}[\sigma^2_{ij}(\mathbf{x}, t''), t'', t']\Big\}\frac{\partial \sigma^1_{ij}(\mathbf{x}, t'')}{\partial t''}\sigma_{ij}(\mathbf{x}, t')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t'.\end{aligned} \tag{2.12}$$

Interchanging the order of integration, and integrating by parts (which is justified in view of the continuity assumption on J and σ_{ij}) the first integral in (2.12) becomes

$$\int_V J[\sigma^2_{ij}(\mathbf{x}, t), t, t]I_2(\mathbf{x}, t)\,\mathrm{d}v - \int_V \int_0^t \frac{\partial J}{\partial t'}[\sigma^2_{ij}(\mathbf{x}, t'), t', t']I_2(\mathbf{x}, t')\,\mathrm{d}t'\,\mathrm{d}v, \tag{2.13}$$

where $I_2(\mathbf{x}, t) = (1/2)\sigma_{ij}(\mathbf{x}, t)\sigma_{ij}(\mathbf{x}, t)$ is the second invariant of the stress tensor. Since $I_2 \geq 0$ we can use the second law of the mean for the first integral in (2.13) to write (2.13) as

$$C_1 \int_V I_2(\mathbf{x}, t)\,\mathrm{d}v - \int_0^t \int_V \frac{\partial J}{\partial t'}[\sigma^2_{ij}(\mathbf{x}, t'), t', t']I_2(\mathbf{x}, t')\,\mathrm{d}v\,\mathrm{d}t', \tag{2.14}$$

where C_1 is a positive constant in view of (2.8) and the continuity assumptions on J, and order of integration has been changed again.

Using the mean value theorem on the term in brackets† in the second integral in (2.12) we can express this second integral as

$$\int_0^t \int_V \frac{\partial J}{\partial \sigma_{kl}}[\bar{\sigma}_{ij}(\mathbf{x}, t'), t', t']\frac{\partial \sigma^1_{ij}(\mathbf{x}, t')}{\partial t'}\sigma_{kl}(\mathbf{x}, t')\sigma_{ij}(\mathbf{x}, t')\,\mathrm{d}v\,\mathrm{d}t', \tag{2.15}$$

where $\bar{\sigma}_{ij}(\mathbf{x}, t')$ is some intermediate value between $\sigma^1_{ij}(\mathbf{x}, t')$ and $\sigma^2_{ij}(\mathbf{x}, t')$.

†Justified in view of the continuity assumptions on J.

Integrating by parts with respect to t'' the third integral in (2.12) can be written as

$$2\int_0^t\int_V \frac{\partial J}{\partial t'}[\sigma_{ij}^2(\mathbf{x},t''),t'',t']|_{t''=t'} I_2(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'$$
$$-\int_0^t\int_V\int_0^{t'}\frac{\partial^2 J}{\partial t''\partial t'}[\sigma_{ij}^2(\mathbf{x},t''),t']\sigma_{ij}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t'')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t'$$

which upon using the second law of the mean for the first integral can be further expressed as

$$2C_2\int_0^t\int_V I_2(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'-\int_0^t\int_V\int_0^{t'}\frac{\partial^2 J}{\partial t''\partial t'}[\sigma_{ij}^2(\mathbf{x},t''),t'',t']\sigma_{ij}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t'')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t' \tag{2.16}$$

where $C_2\geq 0$ in view of assumption (2.9).

Use of the mean value theorem† on the term in brackets in the last integral in (2.12) allows us to write this integral as

$$\int_0^t\int_V\int_0^{t'}\frac{\partial^2 J}{\partial\sigma_{kl}\partial t'}[\hat{\sigma}_{ij}(\mathbf{x},t''),\ t'',t']\,\frac{\partial\sigma_{ij}^1(\mathbf{x},t'')}{\partial t''}\,\sigma_{kl}(\mathbf{x},t'')\sigma_{ij}(\mathbf{x},t')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t', \tag{2.17}$$

where $\hat{\sigma}_{ij}(\mathbf{x},t'')$ is some intermediate value between $\sigma_{ij}^1(\mathbf{x},t'')$ and $\sigma_{ij}^2(\mathbf{x},t'')$.

Substitution of (2.14)–(2.17) in (2.12) leads to

$$\begin{aligned}0&=K(t)+C_1\int_V I_2(\mathbf{x},t)\,\mathrm{d}v-\int_0^t\int_V\frac{\partial J}{\partial t'}[\sigma_{ij}^2(\mathbf{x},t'),t',t']I_2(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'\\&+\int_0^t\int_V\frac{\partial J}{\partial\sigma_{kl}}[\bar{\sigma}_{ij}(\mathbf{x},t'),t',t']\frac{\partial\sigma_{ij}^1(\mathbf{x},t')}{\partial t'}\sigma_{kl}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'\\&+2C_2\int_0^t\int_V I_2(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'-\int_0^t\int_V\int_0^{t'}\frac{\partial^2 J}{\partial t''\partial t'}[\sigma_{ij}^2(\mathbf{x},t''),t'',t']\\&\times\sigma_{ij}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t'')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t'+\int_0^t\int_V\int_0^{t'}\frac{\partial^2 J}{\partial\sigma_{kl}\partial t'}[\hat{\sigma}_{ij}(\mathbf{x},t''),t'',t']\\&\times\frac{\partial\sigma_{ij}^1(\mathbf{x},t'')}{\partial t''}\sigma_{kl}(\mathbf{x},t'')\sigma_{ij}(\mathbf{x},t')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t'\end{aligned}$$

which in view of the non-negativeness of K, I_2, C_1 and C_2 yields

$$\begin{aligned}0&\leq C_1\int_V I_2(\mathbf{x},t)\,\mathrm{d}v+2C_2\int_0^t\int_V I_2(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'\leq\int_0^t\int_V\frac{\partial J}{\partial t'}[\sigma_{ij}^2(\mathbf{x},t'),t',t']\\&\times I_2(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'+\int_0^t\int_V\int_0^{t'}\frac{\partial^2 J}{\partial t''\partial t'}[\sigma_{ij}^2(\mathbf{x},t''),t'',t']\sigma_{ij}(\mathbf{x},t')\\&\times\sigma_{ij}(\mathbf{x},t'')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t'-\int_0^t\int_V\frac{\partial J}{\partial\sigma_{kl}}[\bar{\sigma}_{ij}(\mathbf{x},t'),t',t']\frac{\partial\sigma_{ij}^1(\mathbf{x},t')}{\partial t'}\\&\times\sigma_{kl}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t')\,\mathrm{d}v\,\mathrm{d}t'-\int_0^t\int_V\int_0^{t'}\frac{\partial^2 J}{\partial\sigma_{kl}\partial t'}[\hat{\sigma}_{ij}(\mathbf{x},t''),t'',t']\\&\times\frac{\partial\sigma_{ij}^1(\mathbf{x},t'')}{\partial t''}\sigma_{kl}(\mathbf{x},t'')\sigma_{ij}(\mathbf{x},t')\,\mathrm{d}t''\,\mathrm{d}v\,\mathrm{d}t'.\end{aligned} \tag{2.18}$$

Using the standard inequalities for magnitudes $|a\pm b|\leq|a|+|b|$ and $|\int f|\leq\int|f|$, and the continuity assumptions on J, the r.h.s. of the second inequality in (2.18) is bounded by

†Justified in view of the continuity assumptions on J.

$$C_3\int_0^t\int_V I_2(\mathbf{x},t')\,dv\,dt' + C_4\int_0^t\int_V\int_0^{t'} |\sigma_{ij}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t'')|\,dt''\,dv\,dt'$$
$$+\int_0^t\int_V \left|\frac{\partial J}{\partial\sigma_{kl}}[\bar{\sigma}_{ij}(\mathbf{x},t'),t',t']\left\|\frac{\partial\sigma^1_{ij}(\mathbf{x},t')}{\partial t'}\right\|\sigma_{kl}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t')\right|dv\,dt'$$
$$+\int_0^t\int_V\int_0^{t'} \left|\frac{\partial^2 J}{\partial\sigma_{kl}\partial t'}[\hat{\sigma}_{ij}(\mathbf{x},t''),t'',t']\left\|\frac{\partial\sigma^1_{ij}(\mathbf{x},t'')}{\partial t''}\right\|\sigma_{kl}(\mathbf{x},t'')\sigma_{ij}(\mathbf{x},t')\right|dt''\,dv\,dt' \qquad (2.19)$$

where C_3 and C_4 are positive constants (the existence of which is guaranteed by the continuity assumption on J) such that for $t', t'' \in (0, t)$

$$\left|\frac{\partial J}{\partial t'}[\sigma_{ij}(\mathbf{x},t'),t',t']\right| \le C_3, \qquad \left|\frac{\partial^2 J}{\partial t''\partial t'}[\sigma_{ij}(\mathbf{x},t''),t'',t']\right| \le C_4.$$

Let A_{ij}, B_{ij}, $i, j = 1, 2, 3$, be constants such that

$$\left|\frac{\partial J}{\partial\sigma_{ij}}\right| \le A_{ij}, \qquad \left|\frac{\partial\sigma^1_{ij}}{\partial t}\right| \le B_{ij}.$$

(The existence of such constants is guaranteed by the continuity assumptions on J and σ.) Then, the integrand of the third term in (2.19) is bounded by

$$A_{kl}|\sigma_{kl}|B_{ij}|\sigma_{ij}|,$$

itself bounded by

$$AB\left(\sum_{i,j=1}^{3}|\sigma_{ij}|\right)^2$$

where

$$A = \max_{i,j} A_{ij}, \qquad B = \max_{i,j} B_{ij}.$$

Since via Cauchy's inequality $(\sum_{i,j=1}^{3}|\sigma_{ij}|)^2 \le 9\sum_{i,j=1}^{3}|\sigma_{ij}|^2 = 18I_2$, the third term in (2.19) is bounded by

$$C_5\int_0^t\int_V I_2(\mathbf{x},t')\,dv\,dt'$$

where $C_5 = 18AB$ is a positive constant.

Similarly, invoking the boundedness of $|\partial^2 J(\sigma,t',t)/\partial\sigma_{ij}\partial t|$ (which follows from continuity assumptions), one can bound the last term in (2.19) by

$$C_6\int_0^t\int_V\int_0^{t'}\left|\sum_{k,l=1}^{3}\sigma_{kl}(\mathbf{x},t'')\sum_{i,j=1}^{3}\sigma_{ij}(\mathbf{x},t')\right|dt''\,dv\,dt'$$

where $C_6 > 0$.

Consequently, (2.19) is bounded by

$$C_3\int_0^t\int_V I_2(\mathbf{x},t')\,dv\,dt' + C_4\int_0^t\int_V\int_0^{t'}|\sigma_{ij}(\mathbf{x},t')\sigma_{ij}(\mathbf{x},t'')|\,dt''\,dv\,dt'$$
$$+ C_5\int_0^t\int_V I_2(\mathbf{x},t')\,dv\,dt' + C_6\int_0^t\int_V\int_0^{t'}\left|\sum_{k,l=1}^{3}\sigma_{kl}(\mathbf{x},t'')\sum_{i,j=1}^{3}\sigma_{ij}(\mathbf{x},t')\right|dt''\,dv\,dt'. \qquad (2.20)$$

Making use of the Schwarz inequality corresponding to the inner product $(f, g) = \int_V f_{ij}g_{ij}\,dv$

for second rank tensor fields f, g and interchanging the order of integration, the second term in (2.20) is bounded by

$$C_4 \int_0^t \int_0^{t'} \left(\int_V 2I_2(\mathbf{x}, t')\, dv \right)^{1/2} \left(\int_V 2I_2(\mathbf{x}, t'')\, dv \right)^{1/2} dt''\, dt'.$$

Similarly, recalling $(\Sigma\, \sigma_{ij})^2 \leq 18I_2$, the last term in (2.20) is bounded by

$$C_6 \int_0^t \int_0^{t'} \left(\int_V 18I_2(\mathbf{x}, t')\, dv \right)^{1/2} \left(\int 18I_2(\mathbf{x}, t'')\, dv \right)^{1/2} dt''\, dt'.$$

Thus, from (2.18), (2.20) and the above bounds we obtain, upon setting

$$v^2(t) = \int_V I_2(\mathbf{x}, t)\, dv,$$

$$C_1 v^2(t) + 2C_2 \int_0^t v^2(t')\, dt' \leq (C_3 + C_5) \int_0^t v^2(t')\, dt' + 2(C_4 + 9C_6) \int_0^t \int_0^{t'} v(t')v(t'')\, dt''\, dt'. \tag{2.21}$$

Since

$$\int_0^t \int_0^{t'} v(t')v(t'')\, dt''\, dt' = \frac{1}{2} \left(\int_0^t v(t')\, dt' \right)^2 \leq \frac{1}{2} t \int_0^t v^2(t)\, dt',$$

the last step following from Schwarz' inequality, and since $C_2 \geq 0$, (2.21) yields

$$C_1 v^2(t) \leq [C_3 + C_5 + t(C_4 + 9C_6)] \int_0^t v^2(t')\, dt'. \tag{2.22}$$

Let arbitrary $T > 0$ be given. Then (2.22) implies

$$v^2(t) \leq C \int_0^t v^2(t')\, dt' \tag{2.23}$$

for all $t \in [0, T]$ and where (since $C_1 > 0$)

$$C = [C_3 + C_5 + T(C_4 + 9C_6)]/C_1.$$

Inasmuch as $v^2(0) = 0$, it follows from (2.23) and Gronwall's inequality† that $v^2 \equiv 0$ in $[0, T]$ for any $T > 0$. Thus

$$\sigma_{ij}(\mathbf{x}, t) \equiv 0, \qquad (\mathbf{x}, t) \in \bar{V} \times [0, \infty),$$

i.e. the stresses are unique. The uniqueness of the strains follows from that of the stresses and (2.1) while uniqueness (up to a rigid body displacement) of the displacement vector follows from that of the strains and (2.3), (2.5) and (2.6)–(2.7). This completes the proof of the theorem.

Remark 2.1. Conditions (2.8)–(2.9) as well as the continuity assumptions on the creep function J are sufficient for uniqueness but not necessary, as they are sufficient but not necessary to carry out some of the steps in the proof: integration by parts, use of the mean value theorem, etc.

Remark 2.2. Conditions (2.8) and (2.9) are conditions on the instantaneous response of the material. The conditions that $(\partial J/\partial t)[\sigma_{ij}, t, t]$ and $(\partial J/\partial \sigma_{kl})[\sigma_{ij}, t, t]$ be continuous in σ_{ij} and t, used to carry out the steps leading to (2.13), (2.15) and (2.20) and which are satisfied when J satisfies the continuity hypotheses of the Theorem, are also conditions on the instantaneous

†Details on Gronwall's inequality, as well as the various other inequalities used in this proof, are given in [13].

response of the material. Note that no other conditions are imposed on the instantaneous response which can be linear or nonlinear elastic, include thermal expansion and irradiation swelling (see Section 3), etc.

Remark 2.3. The constitutive relation (2.1) is valid for homogeneous isotropic materials characterized by a single creep function. It is general enough to include many constitutive relations proposed to date (see Section 3).

Remark 2.4. The problem considered is only mildly nonlinear in the sense that only the constitutive relation (2.1) is nonlinear but the field equations and boundary and initial conditions are linear. The above theorem will apply to uncoupled nonlinear thermoviscoelasticity and/or uncoupled irradioviscoelasticity (see Section 3). For the coupled problems however nonlinearity in the field equations would appear and the theorem would not apply.

Remark 2.5. When the constitutive relation (2.1) is linear the theorem provides an alternate uniqueness theorem for linear viscoelasticity. However the proof can be greatly simplified in this case since $J[\sigma_{ij}^1, t't] - J[\sigma_{ij}^2, t't] = J[\sigma_{ij}, t't]$. Incidentally, this last relationship was tacitly and erroneously assumed in the proof of the uniqueness theorem in [9]. The proof given here provides a correct substitute.

3. APPLICATION TO TEMPERATURE AND IRRADIATION INDUCED CREEP

In [11] Cozzarelli and Huang proposed a constitutive relation for thermoirradiation induced creep which includes as a particular case a nonlinear constitutive equation of the isotropic power law type in terms of memory integrals (see eqn (41) in [12]). We first extend this constitutive relation so as to include aging effects through a time hardening procedure similar to that of [12] and thereby obtain an expression in the form of eqn (2.1). Then we specialize the uniqueness theorem of Section 2 to the constitutive relation so obtained thereby obtaining some restrictions on the various material constants appearing in the relation.

The strain–stress relationship to be considered can be written as

$$\epsilon_{ij}(\mathrm{x}, t) = \frac{\partial \xi[\sigma_{ij}(\mathrm{x}, t), t]}{\partial \sigma_{ij}} \tag{3.1}$$

where the energy functional ξ is given by

$$\begin{aligned}\xi = {} & U_{TE} + \sigma_{ij} \int_0^t [\eta_T(t) - \eta_T(t')] \frac{\partial}{\partial t'}\left(\frac{\partial U_{TS}}{\partial \sigma_{ij}}\right) \mathrm{d}t' + \sigma_{ij} \sum_{k=1}^{M} \int_0^t \{1 - \exp[-A_T^{(k)}(\eta_T(t) - \eta_T(t'))]\} \\ & \times \{a_T^{(k)} + (1 - a_T^{(k)}) \exp[-A_T^{(k)}\eta_T(t')]\} \frac{\partial}{\partial t'}\left(\frac{\partial U_{TT}^{(k)}}{\partial \sigma_{ij}}\right) \mathrm{d}t' + \sigma_{ij} \int_0^t [\eta_R(t) - \eta_R(t')] \frac{\partial}{\partial t'}\left(\frac{\partial U_{RS}}{\partial \sigma_{ij}}\right) \mathrm{d}t' \\ & + \sigma_{ij} \sum_{k=1}^{N} \int_0^t \{1 - \exp[-A_R^{(k)}(\eta_R(t) - \eta_R(t'))]\} \\ & \times \{a_R^{(k)} + (1 - a_R^{(k)}) \exp[-A_R^{(k)}\eta_R(t')]\} \frac{\partial}{\partial t'}\left(\frac{\partial U_{RT}^{(k)}}{\partial \sigma_{ij}}\right) \mathrm{d}t'\end{aligned} \tag{3.2}$$

and where

$$U_{TE} = \frac{1 + \nu_E}{E}\left[I_2 - \frac{\nu_E}{2(1 + \nu_E)} I_1^2\right] + (\alpha_T\theta + \alpha_R n)I_1, \tag{3.3a}$$

$$U_{TS} = \frac{C_{TS}}{M_{TS} + 1}\left[I_2 - \frac{\nu_{TS}}{2(1 + \nu_{TS})} I_1^2\right]^{M_{TS}+1}, \tag{3.3b}$$

$$U_{TT}^{(k)} = \frac{C_{TT}^{(k)}}{M_{TT}^{(k)} + 1}\left[I_2 - \frac{\nu_{TT}^{(k)}}{2(1 + \nu_{TT}^{(k)})} I_1^2\right]^{M_{TT}^{(k)}+1}, \quad k = 1, \ldots, M, \tag{3.3c}$$

$$U_{RS} = \frac{C_{RS}}{M_{RS} + 1}\left[I_2 - \frac{\nu_{RS}}{2(1 + \nu_{RS})} I_1^2\right]^{M_{RS}+1}, \tag{3.3d}$$

$$U_{RT}^{(k)} = \frac{C_{RT}^{(k)}}{M_{RT}^{(k)} + 1}\left[I_2 - \frac{\nu_{RT}^{(k)}}{2(1 + \nu_{RT}^{(k)})} I_1^2\right]^{M_{RT}^{(k)}+1}, \quad k = 1, \ldots, N, \tag{3.3e}$$

are respectively the thermoirradioelastic, steady thermal creep, transient thermal creep, steady irradiation creep, transient irradiation creep complementary potentials. In eqns (3.3) $I_1 = \sigma_{ii}$ and $I_2 = (1/2)\sigma_{ij}\sigma_{ij}$ are respectively the first and second invariants of the stress tensor; E is the elastic modulus, ν_E is Poisson's ratio, α_T and α_R are the coefficients of thermal expansion and swelling respectively; θ is the temperature measured from some constant reference and n is the imperfection density increment from some constant reference; C_{TS}, $C_{TT}^{(k)}$'s, C_{RS}, $C_{RT}^{(k)}$'s, M_{TS}, $M_{TT}^{(k)}$'s, M_{RS}, $M_{RT}^{(k)}$'s, ν_{TS}, $\nu_{TT}^{(k)}$'s, ν_{RS} and $\nu_{RT}^{(k)}$'s are material constants, the first four being positive and the next four being non-negative integers.

In eqn (3.2) $a_T^{(k)}$ and $a_R^{(k)}$ are the thermal and irradiation hardening parameters respectively; $A_T^{(k)}$ and $A_R^{(k)}$ are constants the reciprocal of which are analogous to retardation times. Finally, $\eta_T(t)$ and $\eta_R(t)$ are the thermal and irradiation reduced times which account for temperature and flux dependent material properties; they are defined as

$$\eta_T = \int_0^t \Phi_T[T(\tau)]\Psi_T(\dot{n}(\tau)]\,d\tau, \qquad \eta_R = \int_0^t \Phi_R[T(\tau)]\Psi_R[\dot{n}(\tau)]\,d\tau$$

where Φ_R and Φ_T are functions of temperature T, and Ψ_R and Ψ_T are functions of the time rate of change of the imperfection density $\dot{n}$ (and hence of neutron flux).

Substituting (3.3) in (3.2) and (3.2) in (3.1) yields

$$\epsilon_{ij}(\mathrm{x}, t) = \int_0^t J[\sigma_{ij}(\mathrm{x}, t'), t', t]\frac{\partial\sigma_{ij}(\mathrm{x}, t')}{\partial t'}\,dt' \tag{3.4}$$

which is identical to eqn (2.1) with

$$\begin{aligned} J[\sigma_{ij}(\mathrm{x}, t'), t', t] &= \frac{\partial^2 U_{TE}[\sigma_{ij}(\mathrm{x}, t')]}{\partial\sigma_{ij}^2} + [\eta_T(t) - \eta_T(t')]\frac{\partial^2 U_{TS}[\sigma_{ij}(\mathrm{x}, t')]}{\partial\sigma_{ij}^2} \\ &+ \sum_{k=1}^{M}\{1 - \exp[-A_T^{(k)}(\eta_T(t) - \eta_T(t'))]\}\{a_T^{(k)} + (1 - a_T^{(k)})\exp[-A_T^{(k)}\eta_T(t')]\} \\ &\times\frac{\partial^2 U_{TT}^{(k)}[\sigma_{ij}(\mathrm{x}, t')]}{\partial\sigma_{ij}^2} + [\eta_R(t) - \eta_R(t')]\frac{\partial^2 U_{RS}[\sigma_{ij}(\mathrm{x}, t')]}{\partial\sigma_{ij}^2} + \sum_{k=1}^{N}\{1 - \exp[-A_R^{(k)}(\eta_R(t) - \eta_R(t'))]\} \\ &\times\{a_R^{(k)} + (1 - a_R^{(k)})\exp[-A_R^{(k)}\eta_R(t')]\}\frac{\partial^2 U_{RT}^{(k)}[\sigma_{ij}(\mathrm{x}, t')]}{\partial\sigma_{ij}^2} \end{aligned} \tag{3.5}$$

being the thermoirradiation induced creep function. Equation (3.1) with ξ given by (3.2) is an extension of constitutive relation (41) in [11] which includes aging effects through time hardening.

It is now easy to check that J as given by eqn (3.5) satisfies all the hypotheses of the uniqueness theorem of Section 2 provided.

$$\left.\begin{aligned} &-1 < \nu_E, \nu_{RS}, \nu_{TS}, \nu_{RT}, \nu_{TT} \le \frac{1}{2}. \\ &E \ge 0, \\ &A_T^{(k)} \ge 0,\ a_T^{(k)} \ge 0, \qquad k = 1, \ldots, M, \\ &A_R^{(k)} \ge 0,\ a_R^{(k)} \ge 0, \qquad k = 1, \ldots, N, \\ &\frac{d\eta_T}{dt} \ge 0,\ \frac{d\eta_R}{dt} \ge 0. \end{aligned}\right\} \tag{3.6}$$

Remark 4.1. The constants ν_{RS}, ν_{TS}, ν_{RT}, ν_{TT} are analogous to Poisson's ratio ν_E [11]. Consequently the requirement that they be greater than -1 and less than or equal to 1/2 is quite natural.

Remark 4.2. The conditions $a_T^{(k)}$, $a_R^{(k)} \ge 0$ are weaker than the physical conditions $0 \le a_T^{(k)}$, $a_R^{(k)} \le 1$ given in [12].

Remark 4.3. The non decreasingness assumption on η_T and η_R will be satisfied if in particular $\Phi_T\Psi_T \geq 0$ and $\Phi_R\Psi_R \geq 0$, which is observed experimentally[11]. Furthermore, when one introduces a reduced time one would normally expect a one to one correspondance between real time and reduced time[14] and thus one would normally assume that the reduced time is a strictly monotone function of real time.

Finally, we should note that constitutive relations in the form of eqn (2.1) [also (3.4)] are often termed modified (i.e. nonlinear) superposition integrals (with aging included) and arise in continuum mechanical studies of many viscoelastic materials including polymers, concrete and metals at elevated temperatures. Thus one could also easily specialize the uniqueness theorem of Section 2 to a wide variety of other constitutive relations which have been presented in the literature. Such relations were first proposed some years ago (see Leaderman[15], Rabotnov[16] (somewhat different form) and Arutyunyan[17]), although during the last ten years they have received renewed attention (see [11, 12], Schapery[18], Findley and Lai[19], Pipkin and Rogers[20], Rabotnov *et al.*[21], Stouffer[22], Distefano and Todeschini[23] and Rashid[24]).

Acknowledgement—The authors wish to thank the referees for their valuable comments.

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